Math 255B Lecture 23 Notes

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1 Functional Calculus for Bounded Continuous Functions and Bounded Baire Functions

1.1 Approximation in the functional calculus

Let $\varphi, \psi \in C_0(\mathbb{R})$, and let A be self-adjoint. Lsat time, we showed that $\varphi(A)\psi(A) = (\varphi\psi)(A)$ and that $\|\varphi(A)\|_{\mathcal{L}(H,H)} \leq \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}$.

These properties extend to $\varphi, \psi \in C_B(\mathbb{R})$ by approximation (pick $\varphi \in C_0$ with $\varphi_j \to \varphi$ pointwise and boundedly to get $\varphi_j(A) \to \varphi(A)$ weakly). Notice also that if $\varphi_k, \varphi \in C_B$ with $\varphi_j \to \varphi$ pointwise boundedly, then $\varphi_j(A) \to \varphi(A)$ strongly: for all $u \in H$,

$$\begin{aligned}
\|\varphi_j(A)u - \varphi(A)u\|^2 &= \langle (\varphi_j - \varphi)(A)u, (\varphi_j - \varphi)(A)u \rangle \\
&= \int |\varphi_j(\lambda) - \varphi(\lambda)|^2 \, d\mu_n(\lambda) \\
&\to 0
\end{aligned}$$

by dominated convergence. (And if we have uniform approximation, we get operator norm convergence.)

1.2 Domain of A in terms of spectral measures

Recall that if $u \in D(A)$, then $\int \lambda^2 \, d\mu(\lambda) < \infty$ and $\int \lambda^2 \, d\mu(\lambda) = \|Au\|^2$. Assume, conversely, that $\int \lambda^2 \, d\mu_u(\lambda) < \infty$ for $u \in H$. Let $\varphi \in C_0$ with $0 \leq \varphi \leq 1$, and write

$$A\varphi(A)u = \varphi_1(A)u,$$

where $\varphi_1(\lambda) = \lambda\varphi(\lambda)$. We get

$$\begin{aligned}
\|A\varphi(A)u\|^2 &= \langle \varphi_1(A)u, \varphi_1(A)u \rangle \\
&= \int \lambda^2 \varphi(\lambda)^2 \, d\mu_u(\lambda).
\end{aligned}$$

Let $\varphi_j \in C_0$ with $0 \leq \varphi_j \leq 1$ and $\varphi_j \uparrow 1$. Then $\varphi_j(A)u \to u$ in H. Also,

$$\|A(\varphi_j(A) - \varphi_k(A))u\|^2 = \int \lambda^2(\varphi_j^2(\lambda) - \varphi_k^2(\lambda))\, d\mu_u(\lambda) \xrightarrow{j,k\to\infty} 0$$

by dominated convergence, so $A\varphi_j(A)u \to v$ in H. A is closed (as it is self-adjoint), so $u \in D(A)$, and $Au = \lim_{j\to\infty} A\varphi_j(A)u$.

Remark 1.1.

$$\|d\mu_{u,v}\| = \sup_{|\varphi|\leq 1} \underbrace{\left| \int \varphi \, d\mu_{u,v} \right|}_{\langle \varphi(A)u,v\rangle} \leq \|u\| \cdot \|v\|.$$

1.3 Summary of properties of the functional calculus

Let's summarize our results:

Proposition 1.1 (continuous bounded functional calculus)**.** *Let A be self-adjoint. The map $\Phi : C_B(\mathbb{R}) \to \mathcal{L}(H,H)$ sending $\varphi \mapsto \varphi(A)$ has the following properties:*

1. *Φ is an algebra homomorphism.*
2. *$\varphi(A)^* = \overline{\varphi}(A)$.*
3. *$\|\varphi(A)\| \leq \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}$*
4. *For all $u \in H$, the map $d\mu_u : \varphi \mapsto \langle \varphi(A)u, u\rangle$ is a positive measure.*
5. *$1(A) = 1 \in \mathcal{L}(H,H)$.*[1]
6. *If $\varphi_j \to \varphi$ pointwise boundedly, then $\varphi_j(A) \to \varphi(A)$ strongly.*
7. *We have $D(A) = \{u \in H : \int \lambda^2 \, d\mu(\lambda) < \infty\}$, where $Au = \lim_{j\to\infty} A\varphi_j(A)u$ if $\varphi_j \uparrow 1$ and $\varphi_j \in C_0$.*

1.4 Extension of the functional calculus to bounded Baire functions

Next, we extend the functional calculus to the algebra of bounded Baire functions.

Definition 1.1. Let K be a class of functions $f : \mathbb{R} \to \mathbb{C}$. We say that K is **closed under pointwise limits** if for any sequence $f_j \in K$ such that $f = \lim_j f_j$ exists, we have $f \in K$. We let the **Baire functions** $\mathrm{Ba}(\mathbb{R})$ be the smallest class of functions $\mathbb{R} \to \mathbb{C}$ containing $C(\mathbb{R})$ which is closed under pointwise limits.

[1]This actually follows from the fact that Φ is an algebra homomorphism.

Remark 1.2. $\mathrm{Ba}(\mathbb{R})$ is an algebra under pointwise multiplication: Let $f \in C(\mathbb{R})$, and let $K = \{g : fg \in \mathrm{Ba}(\mathbb{R})\}$. $K \supseteq C(\mathbb{R})$ and is closed under pointwise limits, so $K \supseteq \mathrm{Ba}(\mathbb{R})$. Similarly, we extend to $f \in \mathrm{Ba}(\mathbb{R})$.

Remark 1.3. If μ is a positive (Radon) measure and $f \in \mathrm{Ba}(\mathbb{R})$, then f is μ-measurable.

Let $\mathrm{Ba}_b(\mathbb{R})$ be the Banach algebra of **bounded Baire functions** ($\mathrm{Ba}_b(\mathbb{R}) \subseteq L^1(\mu_u)$ for all $u \in H$).

Proposition 1.2. *Let $\varphi \in \mathrm{Ba}_b(\mathbb{R})$. There exists a unique, bounded linear map $\varphi(A) \in \mathcal{L}(H, H)$ such that*

$$\langle \varphi(A)u, v\rangle = \int \varphi(\lambda)\, d\mu_{u,v}(\lambda).$$

Proof. We have to check that $(u, v) \mapsto \int \varphi(\lambda)\, d\mu_{u,v}$ is sesquilinear; the Riesz-representation theorem will provide $\varphi(A)$. We may assume that φ is real, so $|\varphi| \leq M$. Let us check that

$$\int \varphi(\lambda)\, d\mu_{\lambda_1 u_1 + \lambda_2 u_2, v} = \lambda_1 \int \varphi(\lambda)\, d\mu_{u_1,v}(\lambda) + \lambda_2 \int \varphi(\lambda)\, d\mu_{u_2,v}(\lambda).$$

Let $K = \{\varphi \in \mathrm{Ba}_b(\mathbb{R}; \mathbb{R}) : |\varphi| \leq M,$ this condition holds$\}$. Then K contains continuous functions, and K is closed under pointwise limits (by dominated convergence). We need one more claim:

Claim: The class $\{\varphi\, \mathrm{Ba}_b(\mathbb{R}; \mathbb{R}) : |\varphi| \leq M\}$ is the smallest class of functions $\mathbb{R} \to [-M, M]$ containing continuous functions $\mathbb{R} \to [-M, M]$ which is closed under pointwise limits. Check that this holds. We get that $K = \{\varphi \in \mathrm{Ba}_b(\mathbb{R}; \mathbb{R}) : |\varphi| \leq M\}$, and the proposition follows. $\square$